

PROBABILITY THAT A GIVEN ELEMENT OF A GROUP IS A COMMUTATOR OF ANY TWO RANDOMLY CHOSEN GROUP ELEMENTS

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ABSTRACT. We study the probability of a given element, in the commutator subgroup of a group, to be equal to a commutator of two randomly chosen group elements, and compute explicit formulas for calculating this probability for some interesting classes of groups having only two different conjugacy class sizes. We re-prove the fact that if G is a finite group such that the set of its conjugacy class sizes is $\{1, p\}$, where p is a prime integer, then G is isoclinic (in the sense of P. Hall) to an extraspecial p -group.

1. INTRODUCTION

For a finite group G , let $\text{Pr}(G)$ denote the commutativity degree or commuting probability of G , which is defined by

$$\text{Pr}(G) = |\{(x, y) \in G \times G \mid xy = yx\}|/|G|^2,$$

where, for any finite set S , $|S|$ denotes its cardinality. Various aspects of this notion have been studied by many mathematicians over the years. A very impressive historical account can be found in [4, Introduction]. Recently, Pournaki and Sobhani [12] introduced the notion of $\text{Pr}_g(G)$, which is defined by

$$\text{Pr}_g(G) = |\{(x, y) \in G \times G \mid x^{-1}y^{-1}xy = g\}|/|G|^2.$$

Notice that for a given element $g \in G$, $\text{Pr}_g(G)$ measures the probability that the commutator of two randomly chosen group elements is equal to g . Obviously, when $g = 1$, $\text{Pr}_g(G) = \text{Pr}(G)$. Pournaki and Sobhani [12] mainly studied $\text{Pr}_g(G)$ for finite groups G which have only two different irreducible complex character degrees and obtained an impressive formula for $\text{Pr}_g(G)$ for such groups G . In particular, using character theoretic techniques, they obtained explicit formulas for $\text{Pr}_g(G)$, when G is a finite group with $|\gamma_2(G)| = p$, where p is a prime integer and $\gamma_2(G)$ denotes the commutator subgroup of G . In this situation, there can only be two cases, namely, (i) $\gamma_2(G) \leq Z(G)$ or (ii) $\gamma_2(G) \cap Z(G) = 1$, where $Z(G)$ denotes the center of G . These are the cases which were studied by Rusin [13] in 1979 and explicit formulas were obtained for $\text{Pr}(G)$. For finite groups G which have only two different irreducible complex character degrees 1 and m (say), they proved [12, Theorem 2.2] that for each $1 \neq g \in K(G)$,

$$(1.1) \quad \text{Pr}_g(G) = (1/|\gamma_2(G)|)(1 - 1/m^2),$$

where $K(G)$ denotes the set of all commutators in G . We assume that $1 \in K(G)$.

Motivated by the results of Rusin [13], and Pournaki and Sobhani [12], we investigate $\text{Pr}_g(G)$ for some classes of finite groups G which have only two different conjugacy class sizes. Explicit formulas are obtained for some interesting classes of finite groups using purely group theoretic techniques. A finite group G is said to be a Camina group if $x^G = x\gamma_2(G)$ for all

2000 *Mathematics Subject Classification.* Primary 20D60; Secondary 20P05.

Key words and phrases. Camina group, conjugacy class size, isoclinism of groups.

$x \in G - \gamma_2(G)$. It was proved by Dark and Scoppola [1] that the nilpotency class of a finite Camina p -group is at most 3. For such groups G of nilpotency class 2, we obtain a very nice formula for $\text{Pr}_g(G)$ in the following theorem, proof of which follows from Theorem 3.3. Explicit formulas for such groups of class 3 are obtained in Section 5.

Theorem A. *Let G be a finite Camina p -group of nilpotency class 2 such that $|\gamma_2(G)| = p^r$, where p is a prime number and r is a positive integer. Then, for $g \in K(G)$,*

$$\text{Pr}_g(G) = \begin{cases} \frac{1}{p^r} \left(1 + \frac{p^r - 1}{p^{2m}}\right) & \text{if } g = 1 \\ \frac{1}{p^r} \left(1 - \frac{1}{p^{2m}}\right) & \text{if } g \neq 1, \end{cases}$$

for some positive integer m such that $r < m$.

Denote by $P(G)$ the set $\{\text{Pr}_g(G) \mid 1 \neq g \in K(G)\}$. It follows from [12, Theorem 2.2] (see (1.1)) that $P(G)$ is a singleton set for all finite groups G which have only two different irreducible complex character degrees. It follows from Theorem A that $P(G)$ is a singleton set for all finite Camina p -groups G of nilpotency class 2. Notice that finite Camina p -groups of class 2 forms a subclass of finite p -groups having only two different conjugacy class sizes. Are there also other classes of finite p -groups G of class 2 having only two different conjugacy class sizes and $|P(G)| = 1$? The answer is affirmative as we show in the following theorem, which we prove in Section 4.

Theorem B. *For any positive integer $r \geq 1$, there exists a group G of order $p^{(r+1)(r+2)/2}$ such that $|P(G)| = 1$. Moreover, $\text{Pr}_g(G) = (p^2 - 1)/p^{2r+1}$ for each $1 \neq g \in K(G)$, and if $r > 1$, G is not a Camina group.*

Let G be an arbitrary finite group having only two different conjugacy class sizes. Ito [7] proved that such a group G is a prime power order group (upto abelian direct factors). It was then proved by Ishikawa [6] that the nilpotency class of such a group G is bounded above by 3. It follows from Theorem 2.3 (see below) that for studying $\text{Pr}_g(G)$ for arbitrary finite groups having only two different conjugacy class sizes, it is sufficient to study it for such p -groups only. Thus by Ishikawa's result, it is sufficient to consider such p -groups of nilpotency class 2 and 3 only. In Section 5, we show that there exists a finite p -group G of class 3 having only two different conjugacy class sizes and $|P(G)| > 1$. But, to the best of our knowledge, no example of a finite p -group G of class 2 having only two different conjugacy class sizes and $|P(G)| \neq 1$ is known. This gives rise to the following natural question.

Question. Is it true that $|P(G)| = 1$ for all finite p -groups G of class 2 having only two different conjugacy class sizes?

In Section 5, we also study some bounds on $\text{Pr}_g(G)$ for a finite group G . K. Ishikawa [5] proved that if G is any finite group having only two different conjugacy class sizes 1 and p , where p is a prime integer, then G is isoclinic (see Section 2 for the definition) to an extraspecial p -group. This makes use of a very deep result of Vaughan-Lee [9, Main Theorem], which was proved using Lie theoretic arguments. We re-prove this statement using elementary arguments on commuting probability.

In the light of above discussion, it is interesting to pose the following natural problem.

Problem. Classify all finite groups G such that $|P(G)| = 1$.

This problem can also be viewed in the following manner. Let $\alpha : G \times G \rightarrow \gamma_2(G)$ be a commutator map defined by $\alpha(g_1, g_2) = [g_1, g_2]$ for each pair $(g_1, g_2) \in G \times G$. Let, for $g \in K(G)$, $n_g(G)$ denote the set $\alpha^{-1}(g)$. So $n_g(G)$, in some sense, can be viewed as a fiber on $g \in K(G)$ in $G \times G$. Now suppose that $N(G) := \{|n_g(G)| \mid 1 \neq g \in K(G)\}$. Notice that $\text{Pr}_g(G) = |n_g(G)|/|G|^2$. Thus classifying group G with $|P(G)| = 1$ is nothing but classifying groups with $|N(G)| = 1$. Such things are defined and studied in [14] for some other purposes.

2. PRELIMINARY RESULTS

In 1940, P. Hall [3] introduced the following concept of isoclinism on the class of all groups.

Let X be a finite group and $\bar{X} = X/Z(X)$. Then commutation in X gives a well defined map $a_X : \bar{X} \times \bar{X} \rightarrow \gamma_2(X)$ such that $a_X(xZ(X), yZ(X)) = [x, y]$ for $(x, y) \in X \times X$. Two finite groups G and H are called *isoclinic* if there exists an isomorphism α of the factor group $\bar{G} = G/Z(G)$ onto $\bar{H} = H/Z(H)$, and an isomorphism β of the subgroup $\gamma_2(G)$ onto $\gamma_2(H)$ such that the following diagram is commutative

$$(2.1) \quad \begin{array}{ccc} \bar{G} \times \bar{G} & \xrightarrow{a_G} & \gamma_2(G) \\ \alpha \times \alpha \downarrow & & \downarrow \beta \\ \bar{H} \times \bar{H} & \xrightarrow{a_H} & \gamma_2(H). \end{array}$$

The resulting pair (α, β) is called an *isoclinism* of G onto H . Notice that isoclinism is an equivalence relation among finite groups.

The following result is from [3].

Theorem 2.2. *Let G be any group. Then there exists a group H such that G and H are isoclinic and $Z(H) \leq \gamma_2(H)$.*

Pournaki and Sobhani [12] proved the following interesting result. Since the result is very important for our study, we provide a brief proof here.

Theorem 2.3. *Let G and H be two isoclinic finite groups with isoclinism (α, β) . Then $\text{Pr}_g(G) = \text{Pr}_{\beta(g)}(H)$.*

Proof. Since (α, β) is an isoclinism from G onto H , diagram (2.1) commutes. Let $g \in \gamma_2(G)$ be the given element. Consider the sets $S_g = \{(g_1Z(G), g_2Z(G)) \in G/Z(G) \times G/Z(G) : [g_1, g_2] = g\}$ and $S_{\beta(g)} = \{(h_1Z(H), h_2Z(H)) \in H/Z(H) \times H/Z(H) : [h_1, h_2] = \beta(g)\}$. Since the above diagram commutes, it follows that $|S_g| = |S_{\beta(g)}|$. Since the maps a_G and a_H are well defined, we have $|\{(g_1, g_2) \in G \times G : [g_1, g_2] = g\}| = |Z(G)|^2 |S_g|$ and $|\{(h_1, h_2) \in H \times H : [h_1, h_2] = \beta(g)\}| = |Z(H)|^2 |S_{\beta(g)}|$. Notice that $|G : Z(G)| = |H : Z(H)|$. Hence

$$\text{Pr}_g(G) = \frac{|S_g|}{|G : Z(G)|^2} = \frac{|S_{\beta(g)}|}{|H : Z(H)|^2} = \text{Pr}_{\beta(g)}(H).$$

This completes the proof.

We remark that Theorem 2.3 was proved by P. Lescot [8, Lemma 2.4] for $g = 1$.

Now we prove the following interesting result.

Proposition 2.4. *Let G be a finite nilpotent group such that $\gamma_2(G)$ is finite of prime power order of some prime integer p . Then G is isoclinic to a finite p -group H such that $Z(H) \leq \gamma_2(H)$.*

Proof. By Theorem 2.2 there exists a group H isoclinic to G such that $Z(H) \leq \gamma_2(H) \cong \gamma_2(G)$. Since G is nilpotent, H is nilpotent and therefore it can be written as a direct sum of its Sylow p -subgroups. We claim that H is a p -group, where p is the prime integer given in the statement. Suppose that q is a prime integer not equal to p and q divides $|H|$. Then H has a Sylow q -subgroup Q (say) such that $\gamma_2(Q)$ is a q -group contained in $\gamma_2(G)$. But, by our supposition, $\gamma_2(G)$ is a p -group. Thus $\gamma_2(Q)$ must be trivial. This shows that Q is abelian and therefore $Q \leq Z(H) \leq \gamma_2(H)$, which is a contradiction to the fact that $\gamma_2(H) \cong \gamma_2(G)$ is a p -group. Hence our claim is true and the proof is complete. \square

A p -group G is said to be *special* if $Z(G) = \gamma_2(G) = \Phi(G)$, where $\Phi(G)$ denotes the Frattini subgroup of G . A special p -group G is called *extraspecial* if the order of $Z(G)$ is p . Notice that a p -group G is extraspecial if $Z(G) = \gamma_2(G)$ is of order p .

Corollary 2.5. *Let G be a finite group such that $|\gamma_2(G)| = p$ and $\gamma_2(G) \leq Z(G)$, where p is a prime integer. Then G is isoclinic to an extraspecial p -group.*

The following theorem follows from [7], [6] and the definition of isoclinism.

Theorem 2.6. *Let G be a finite group having only two different conjugacy class sizes 1 and m (say). Then G is isoclinic to a finite p -group of class at most 3 for some prime integer p . Moreover, $m = p^r$ for some positive integer r .*

The following result is due to I. D. Macdonald [10, Corollary 2.4].

Theorem 2.7. *Any finite Camina p -group of nilpotency class 2 is special.*

3. PROOF OF THEOREM A

For a group G and any element $x \in G$, $[x, G]$ denotes the set $\{[x, g] = x^{-1}g^{-1}xg \mid g \in G\}$. We start with the following extremely useful expression for $\text{Pr}_g(G)$ from [2]. We mention a slightly modified proof here.

Lemma 3.1. *Let G be a finite group and $g \in G$. Then*

$$\text{Pr}_g(G) = \frac{1}{|G|} \sum_{g \in [x, G]} \frac{1}{|x^G|}.$$

Proof. Notice that $\{(x, y) \in G \times G : [x, y] = g\} = \bigcup_{x \in G} (\{x\} \times T_x)$, where $T_x = \{y \in G : [x, y] = g\}$. Further notice that, for any $x \in G$, the set T_x is non-empty if and only if $xg \in x^G$. Suppose that T_x is non-empty for some $x \in G$. Fix an element $t \in T_x$. It is easy to see that $T_x = tC_G(x)$. The proof of the lemma now follows from the definition of $\text{Pr}_g(G)$. \square

For a finite group G , by $b(G)$ we denote the size of the largest conjugacy class in G . Notice that $b(G) \leq |\gamma_2(G)|$. For $g = 1$, the preceding lemma gives the following result.

Proposition 3.2. *Let G be a finite group. Then*

$$\text{Pr}(G) \geq \frac{1}{|b(G)|} \left(1 + \frac{|b(G)| - 1}{|G : Z(G)|} \right).$$

Moreover,

- (i) Equality holds if and only if $b(G) = |x^G|$ for all $x \in G - Z(G)$;
- (ii) If G is non-abelian, then $\text{Pr}(G) > \frac{1}{|b(G)|}$.

Theorem 3.3. *Let G be a finite nilpotent group of class 2 such that $|\gamma_2(G)| = p^r$ and $[x, G] = \gamma_2(G)$ for all $x \in G - Z(G)$, where p is a prime and r is a positive integer. Then*

$$\Pr_g(G) = \begin{cases} \frac{1}{p^r} \left(1 + \frac{p^r - 1}{p^{2m}}\right) & \text{if } g = 1 \\ \frac{1}{p^r} \left(1 - \frac{1}{p^{2m}}\right) & \text{if } g \neq 1, \end{cases}$$

for some positive integer m . Moreover, m is independent of the choice of G in its isoclinism family.

Proof. Let G be a group as given in the statement. Then it follows from Proposition 2.4 that G is isoclinic to a Camina p -group H (say) of class 2. Suppose that (α, β) is an isoclinism between G and H . It follows from [10, Theorem 3.2] that $|H/Z(H)| = p^{2m}$ for some positive integer m . Now by Theorem 2.3, it follows that $\Pr_g(G) = \Pr_{\beta(g)}(H)$. So, now onwards we work with H only. Notice that $x^H = x\gamma_2(H)$ for all $x \in H - Z(H)$. Thus G has only two conjugacy class sizes, namely, 1 and $|\gamma_2(G)|$ and therefore $b(H) = |\gamma_2(H)| = p^r$ for all $x \in H - Z(H)$. If $h = 1$, then it follows from Proposition 3.2 that

$$\Pr_h(H) = \frac{1}{p^r} \left(1 + \frac{p^r - 1}{|H : Z(H)|}\right).$$

Now assume that $h \neq 1$. Notice that if $x \in Z(H)$ and $h \in [x, H]$, then $h = 1$. Since $h \in [x, H]$ and $|x^H| = b(H)$ for all $x \in H - Z(H)$, by Lemma 3.1 we have

$$\begin{aligned} \Pr_h(H) &= \frac{1}{|H|} \sum_{x \in H - Z(H)} \frac{1}{|x^H|} = \frac{|H - Z(H)|}{|b(H)||H|} \\ &= \frac{1}{b(H)} \left(1 - \frac{1}{|H : Z(H)|}\right). \end{aligned}$$

Since $|H : Z(H)| = p^{2m}$, we have

$$\Pr_h(H) = \begin{cases} \frac{1}{p^r} \left(1 + \frac{p^r - 1}{p^{2m}}\right) & \text{if } h = 1 \\ \frac{1}{p^r} \left(1 - \frac{1}{p^{2m}}\right) & \text{if } h \neq 1. \end{cases}$$

Since β is an isomorphism from $\gamma_2(G)$ onto $\gamma_2(H)$, for each $g \in \gamma_2(G)$, there exists an $h \in \gamma_2(H)$ such that $h = \beta(g)$. The fact that $|G_1/Z(G_1)| = |G_2/Z(G_2)|$ for any two isoclinic finite groups G_1 and G_2 shows that the integer m appeared above is independent of the choice of G in its isoclinism family. This completes the proof of the theorem. \square

Let G be a finite group and H be its normal subgroup of order p , where p is a prime dividing $|G|$ such that $(|G|, p-1) = 1$. Then it follows from N/C lemma (i.e., $N_G(H)/C_G(H)$ embeds in $\text{Aut}(H)$) that H is a central subgroup of G . As a corollary of the preceding theorem, we get the following interesting result.

Corollary 3.4. *Let G be a finite group with $|\gamma_2(G)| = p$, where p is a prime integer. If either G is nilpotent or p is a prime dividing $|G|$ such that $(|G|, p-1) = 1$, then*

- (i) G is isoclinic to an extraspecial p -group;
- (ii) for $g \in \gamma_2(G)$, we have

$$\Pr_g(G) = \begin{cases} \frac{1}{p} \left(1 + \frac{p-1}{p^{2m}}\right) & \text{if } g = 1 \\ \frac{1}{p} \left(1 - \frac{1}{p^{2m}}\right) & \text{if } g \neq 1, \end{cases}$$

for some positive integer m .

Proof. Notice that $\gamma_2(G)$ is a central subgroup of G in both of the cases. Thus G is a nilpotent group of class 2 and therefore, by Corollary 2.5, it is isoclinic to an extraspecial p -group. Now the result follows from Theorem 3.3 for $r = 1$. \square

In the following result we characterize (upto isoclinism) finite groups G such that $\text{Pr}(G) = \frac{1}{|b(G)|} \left(1 + \frac{|b(G)|-1}{|G:Z(G)|}\right)$.

Proposition 3.5. *Let G be a non-abelian finite group such that $\text{Pr}(G) = \frac{1}{|b(G)|} \left(1 + \frac{|b(G)|-1}{|G:Z(G)|}\right)$. Then G is isoclinic to a finite p -group having only two different conjugacy class sizes 1 and p^r for some prime p and positive integer r . Moreover, if $b(G) = |\gamma_2(G)|$, then G is isoclinic to a Camina special p -group.*

Proof. Let G be the group as in the statement. Then it follows from Proposition 3.2(i) that G has only two conjugacy class sizes, namely, 1 and $b(G)$. Now it follows from Theorem 2.6 that G is isoclinic to a finite p -group of nilpotency class at most 3 for some prime integer p and $b(G) = p^r$ for some positive integer r . Thus the first assertion of the statement holds true.

Now assume that $b(G) = |\gamma_2(G)|$. We claim that the nilpotency class of G is at most 2. Suppose that the nilpotency class is 3. Then there exists an element $u \in \gamma_2(G) - Z(G)$. Then $1 \neq |u^G| = |[u, G]| \leq |\gamma_3(G)| < |\gamma_2(G)| = b(G)$, which is a contradiction to the fact that G has only two conjugacy class sizes. Since G is non-abelian, it now follows that the nilpotency class of G is 2. As (by Theorem 2.2) there exists a finite p -group H such that G and H are isoclinic and $Z(H) \leq \gamma_2(H)$, it follows that H is a Camina group of nilpotency class 2. That H is special, now follows from Theorem 2.7. This completes the proof. \square

4. PROOF OF THEOREM B

For any positive integer $r \geq 1$, consider the following group constructed by Ito [7].

$$(4.1) \quad G = \langle x_1, \dots, x_{r+1} \mid [x_i, x_j] = y_{ij}, [x_k, y_{ij}] = 1, \\ x_i^p = x_{r+1}^p = y_{ij}^p, 1 \leq i < j \leq r+1 \rangle.$$

Some interesting properties of this group are given in the following lemma, proof of which follows from [7, Example 1].

Lemma 4.2. *The group G defined in (4.1) is a special p -group of order $p^{(r+1)(r+2)/2}$ and exponent p , and $|\gamma_2(G)| = p^{r(r+1)/2}$. This group has only two different conjugacy class sizes, namely 1 and p^r .*

We now proceed to prove Theorem B. We start with the following technical result, which is also of independent interest.

Lemma 4.3. *Let G be a finite p -group of nilpotency class 2 minimally generated by x_1, \dots, x_d such that the exponent of $\gamma_2(G)$ is p . Let $g_1 = \prod_{i=1}^d x_i^{\alpha_i}$ and $g_2 = \prod_{j=1}^d x_j^{\beta_j}$ be two different elements of G such that $[g_1, g_2] \neq 1$, where α_i, β_j are some non negative integers between 0 and $p-1$. Then we can find another minimal generating set for G containing g_1, g_2 .*

Proof. Let k and l be the smallest integers such that $\alpha_k \neq 0$ and $\beta_l \neq 0$ respectively. First suppose that $k \neq l$. Assume that $k < l$. Then it is not difficult to see that the set

$$\{x_1, \dots, x_{k-1}, g_1, x_{k+1}, \dots, x_{l-1}, g_2, x_{l+1}, \dots, x_d\}$$

minimally generates G . Now suppose that $k = l$. If $\alpha_l = \beta_k$, then choose the next smallest m such that $\alpha_m \neq \beta_m$. Let t denote the largest integer $< m$ such that $\alpha_t = \beta_t$. Then it follows that if we replace x_t by g_1 and x_m by g_2 in the given generating set, we'll again get a minimal generating set. If $\alpha_l \neq \beta_k$, then the following three cases may occur:

- (i) There exists an integer s , $k \leq s \leq d$, which is the smallest such that $\alpha_s = \beta_s \neq 0$.
- (ii) There exists an integer s , $k \leq s \leq d$, which is the smallest such that $\alpha_s = 0$, $\beta_s \neq 0$ or $\alpha_s \neq 0$, $\beta_s = 0$.
- (iii) For each $k \leq s \leq d$, $\alpha_s \neq 0$ if and only if $\beta_s \neq 0$, and $\alpha_s \neq \beta_s$ if these are different from zero.

In the situation of case (i), we can get a required minimal generating set by replacing x_{s-1} by g_1 and x_s by g_2 in the given generating set. If case (ii) occurs, then choose the largest $t < s$ such that $\alpha_t \neq 0$, $\beta_t \neq 0$. So by replacing x_t by g_1 and x_s by g_2 in the given generating set, we get a required minimal generating set.

Now assume the last case, i.e., for each $1 \leq s \leq d$, $\alpha_s \neq 0$ if and only if $\beta_s \neq 0$, and $\alpha_s \neq \beta_s$ if these are different from zero. Let us re-write g_1 and g_2 by omitting x_i 's for which $\alpha_i = \beta_i = 0$. So $g_1 = \Pi x_{e_i}^{\alpha_{e_i}}$ and $g_2 = \Pi x_{f_j}^{\beta_{f_j}}$, where e_i and f_j occur in the increasing order. In the rest of the proof, we confine all of our computations in \mathbb{F}_p , the field consisting of p elements.

Suppose that there exists some positive integer t such that $\alpha_{e_t}\beta_{e_{t+1}} \neq \alpha_{e_{t+1}}\beta_{e_t}$. Consider the set

$$X := \{x_1, \dots, x_{e_t-1}, g_1, x_{e_t+1}, \dots, x_{e_{t+1}-1}, g_2, x_{e_{t+1}+1}, \dots, x_d\}.$$

It is easy to see that $y_1 := x_{e_t}^{\alpha_{e_t}} x_{e_{t+1}}^{\alpha_{e_{t+1}}}$ and $y_2 := x_{e_t}^{\beta_{e_t}} x_{e_{t+1}}^{\beta_{e_{t+1}}}$ can be produced by elementary cancelations in the set X . At this stage, we can have two possibilities, namely (a) $[x_{e_t}, x_{e_{t+1}}] \neq 1$ or (b) $[x_{e_t}, x_{e_{t+1}}] = 1$. First assume (a), i.e., $[x_{e_t}, x_{e_{t+1}}] \neq 1$. Since $\alpha_{e_t}\beta_{e_{t+1}} \neq \alpha_{e_{t+1}}\beta_{e_t}$, we claim that the elements y_1 and y_2 can not commute with each other. Indeed, if $[y_1, y_2] = 1$, then by a straight forward calculation it follows that

$$[x_{e_t}, x_{e_{t+1}}]^{\alpha_{e_t}\beta_{e_{t+1}} - \alpha_{e_{t+1}}\beta_{e_t}} = 1.$$

Since the exponent of $\gamma_2(G)$ is p , this is possible only when $\alpha_{e_t}\beta_{e_{t+1}} = \alpha_{e_{t+1}}\beta_{e_t}$, which contradicts our supposition. Hence our claim follows. Consider the subgroup $H = \langle x_{e_t}, x_{e_{t+1}} \rangle$. Since the nilpotency class of G is 2 and $[x_{e_t}, x_{e_{t+1}}] \neq 1$, H is a non-abelian group of class 2. Obviously $y_1, y_2 \in H - \Phi(H)$, where $\Phi(H)$ denotes the Frattini subgroup of H . Since $[y_1, y_2] \neq 1$, it follows that y_1 and y_2 generate H . This proves that X generates G and therefore X is a minimal generating set for G .

Now assume (b), i.e., $[x_{e_t}, x_{e_{t+1}}] = 1$. Let γ_{e_t} and δ_{e_t} be the multiplicative inverses of α_{e_t} and β_{e_t} in \mathbb{F}_p respectively. Notice that

$$y_1^{\gamma_{e_t}} y_2^{-\delta_{e_t}} = x_{e_{t+1}}^{\alpha_{e_{t+1}}\gamma_{e_t} - \beta_{e_{t+1}}\delta_{e_t}}.$$

We claim that $\alpha_{e_{t+1}}\gamma_{e_t} - \beta_{e_{t+1}}\delta_{e_t}$ not equal to zero in \mathbb{F}_p . If possible, assume the contrary. Thus $\alpha_{e_{t+1}}\gamma_{e_t} = \beta_{e_{t+1}}\delta_{e_t}$. Multiplying both sides by $\alpha_{e_t}\beta_{e_t}$, we get $\alpha_{e_t}\beta_{e_{t+1}} = \alpha_{e_{t+1}}\beta_{e_t}$, since $\alpha_{e_t}\gamma_{e_t} = 1 = \beta_{e_t}\delta_{e_t}$. This gives a contradiction to our supposition. Hence our claim is true.

Now it is easy to see that y_1 and y_2 generate the abelian subgroup $\langle x_{e_t}, x_{e_{t+1}} \rangle$. Hence X is a minimal generating set for G .

Finally assume that $\alpha_{e_t}\beta_{e_{t+1}} = \alpha_{e_{t+1}}\beta_{e_t}$ for all possible positive integers t . In this case, we are going to show that $[g_1, g_2] = 1$, which is not possible by the given hypothesis and therefore this case does not occur. Let t' and t'' be two arbitrary integers such that $1 \leq t' < t'' \leq d'$. Then there exist integers t_1, \dots, t_r (say) such that $t' = t_1 < \dots < t_r = t''$. We now have the following system of equations.

$$\alpha_{e_{t_i}}\beta_{e_{t_{i+1}}} = \alpha_{e_{t_{i+1}}}\beta_{e_{t_i}}, \quad 1 \leq i \leq r-1.$$

Solving this system of equations, we get $\alpha_{e_{t_1}}\beta_{e_{t_r}} = \alpha_{e_{t_r}}\beta_{e_{t_1}}$. This shows that

$$[x_{e_{t'}}^{\alpha_{e_{t'}}}, x_{e_{t''}}^{\beta_{e_{t''}}}] [x_{e_{t''}}^{\alpha_{e_{t''}}}, x_{e_{t'}}^{\beta_{e_{t'}}}] = [x_{e_{t'}}, x_{e_{t''}}]^{\alpha_{e_{t'}}\beta_{e_{t''}} - \alpha_{e_{t''}}\beta_{e_{t'}}} = 1.$$

Since t' and t'' were arbitrary, it follows that $[g_1, g_2] = 1$. This completes the proof of the lemma. \square

Lemma 4.4. *Let G be the p -group defined in (4.1), where p is an odd prime. Then $|\{x \in G \mid y_{ij} \in [x, G]\}| = (p^2 - 1)p^{r(r+1)/2}$.*

Proof. First we claim that $|\{x \in G \mid y_{ij} \in [x, G]\}|$ is at least $(p^2 - 1)p^{r(r+1)/2}$. Consider the subgroup Y_{ij} of G generated by x_i, x_j . Notice that Y_{ij} is a group of order p^3 and exponent p . So it follows that $y_{ij} \in [w, Y_{ij}]$ for all $w \in Y_{ij} - \langle y_{ij} \rangle$. Since the order of $\gamma_2(G) = Z(G)$ is $p^{r(r+1)/2}$, it follows that $|\{w \in Y_{ij}\gamma_2(G) \mid y_{ij} \in [w, Y_{ij}\gamma_2(G)]\}| = (p^2 - 1)p^{r(r+1)/2}$. This proves our claim.

Now we show that $|\{x \in G \mid y_{ij} \in [x, G]\}|$ is not more than $(p^2 - 1)p^{r(r+1)/2}$. Suppose that there exists a pair of elements (g_1, g_2) in G such that $y_{ij} = [g_1, g_2]$ and at least one of g_1, g_2 lie(s) outside $Y_{ij}\gamma_2(G)$. Let $g_1 = x_i^{\alpha_i} x_j^{\alpha_j} \prod_{k \neq i, j} x_k^{\alpha_k} u_1$ and $g_2 = x_i^{\beta_i} x_j^{\beta_j} \prod_{l \neq i, j} x_l^{\alpha_l} u_2$ for some non-negative integers $\alpha_i, \alpha_j, \beta_i, \beta_j, \alpha_k, \beta_l$, and some $u_1, u_2 \in \gamma_2(G)$. Suppose that g_1 lies outside $Y_{ij}\gamma_2(G)$, and $x_k^{\alpha_k}$ and $x_k^{\beta_k}$ appears in g_1 and g_2 respectively with $\alpha_k \neq 0$. First suppose that $\beta_k = 0$. Consider the commutator $[x_i^{\alpha_i} x_j^{\alpha_j} x_k^{\alpha_k}, x_i^{\beta_i} x_j^{\beta_j}]$, which after expanding becomes $[x_i, x_j]^{\alpha_i \beta_j - \alpha_j \beta_i} [x_k, x_i]^{\alpha_k \beta_i} [x_k, x_j]^{\alpha_k \beta_j}$. Notice that the preceding expression is a part of $[g_1, g_2] = y_{ij} = [x_i, x_j]$. Since $\gamma_2(G)$ is independently generated by all y_{rs} , comparing powers of corresponding y_{rs} we get

$$\alpha_i \beta_j - \alpha_j \beta_i = 1, \quad \alpha_k \beta_i = 0, \quad \alpha_k \beta_j = 0.$$

Since $\alpha_k \neq 0$ and we are in a field, it follows that both β_i and β_j are 0. Hence $0 = \alpha_i \beta_j - \alpha_j \beta_i = 1$, which is not possible. This shows that β_k can not be 0 whenever $\alpha_k \neq 0$.

Now assume that both α_k as well as β_k are non zero. It is not difficult to get similar kind of contradiction by considering the commutator $[x_i^{\alpha_i} x_j^{\alpha_j} x_k^{\alpha_k}, x_i^{\beta_i} x_j^{\beta_j} x_k^{\beta_k}]$ and comparing corresponding powers of y_{rs} . This shows that no pair (g_1, g_2) in G lying outside $Y_{ij}\gamma_2(G)$ can give $y_{ij} = [g_1, g_2]$. This completes the proof of the lemma. \square

Now we are ready to prove Theorem B.

Proof of Theorem B. Let G be the group defined in (4.1). Let $1 \neq g \in K(G)$. Since the nilpotency class of G is 2 and the exponent of G is p (notice that we only need the fact that the exponent of $G/Z(G)$ is p), there exist $g_1, g_2 \in G$ such that $g_1 = \prod_{i=1}^d x_i^{\alpha_i}$, $g_2 = \prod_{j=1}^d x_j^{\beta_j}$

and $1 \neq g = [g_1, g_2]$, where α_i and β_j are some non negative integers between 0 and $p-1$. By Lemma 4.3, we can find a minimal generating set $\{w_1, \dots, w_d\}$ for G which includes both g_1 and g_2 . Notice that $w_i^p = 1$ for $1 \leq i \leq d$ and $\gamma_2(G)$ is minimally generated by $[w_i, w_j]$, where $[w_i, w_j]$ are central elements of order p for $1 \leq i < j \leq d$. Since G has only two different conjugacy class sizes 1 and p^r (Lemma 4.2), we have $|x^G| = b(G) = p^r$ for all $x \in G - Z(G)$. Thus by Lemma 4.4, it follows that $|\{x \in G \mid g \in [x, G]\}| = (p^2 - 1)p^{r(r+1)/2}$. Hence

$$\text{Pr}_g(G) = \frac{1}{|G|p^r}(p^2 - 1)p^{r(r+1)/2} = \frac{p^2 - 1}{p^{2r+1}},$$

since $|G| = p^{(r+1)(r+2)/2}$. Obviously $\text{Pr}_g(G)$ is independent of g and therefore $|P(G)| = 1$.

Notice that for $r = 1$, G is a non-abelian group of order p^3 . Thus it is an extraspecial p -group and therefore a Camina group. But if $r \geq 2$, then $|\gamma_2(G)| = p^{r(r+1)/2} > p^r = b(G)$. This shows that G is not a Camina group, because in a finite Camina G , $|\gamma_2(G)| = b(G)$. This completes the proof. \square

The following result follows from [6, Corollary 2.2].

Lemma 4.5. *Let G be a finite p -group of nilpotency class 2 having only two different conjugacy class sizes. Then both $G/Z(G)$ as well as $\gamma_2(G)$ are elementary groups.*

In the following result, we show that $P(H) = 1$ for all p -groups H of class 2 having only two different conjugacy class sizes and having the same commutator structure as of the group G defined in (4.1).

Theorem 4.6. *Let H be a finite p -groups of nilpotency class 2 having only two different conjugacy class sizes. Further, let H be minimally generated by $\{w_1, w_2, \dots, w_d\}$ such that $|\gamma_2(H)| = p^{d(d-1)/2}$. Then $P(H) = 1$.*

Proof. Notice that $Z(H) = \Phi(H)$. For, it follows from the preceding lemma that the exponent of $H/Z(H)$ is p . Thus $H^p \leq Z(H)$. Since H is minimally generated by d elements and $|\gamma_2(H)| = p^{d(d-1)/2}$, no central element can lie in $H - \Phi(H)$. Now using the fact that $\gamma_2(H) \leq Z(H)$, it follows that $Z(H) = \Phi(H)$. By the preceding lemma it follows that $\gamma_2(H)$ is elementary abelian p -groups. Let G be the group defined in (4.1) with $r = d - 1$. Notice that $G/Z(G)$ as well as $\gamma_2(G)$ are elementary abelian and $Z(G) = \Phi(G)$ (this may be observed directly from the presentation of the group or by using Lemma 4.5 and the above information as G satisfies the conditions of Lemma 4.5). By the given hypothesis $|G/Z(G)| = |H/Z(H)|$ and $|\gamma_2(G)| = |\gamma_2(H)|$. Thus $G/Z(G) \cong H/Z(H)$ and $\gamma_2(G) \cong \gamma_2(H)$. Since $|\gamma_2(H)| = p^{d(d-1)/2}$, $[w_i, w_j] \neq 1$ for all $1 \leq i < j \leq d$. Set $[w_i, w_j] = z_{ij}$. Notice that the map $\alpha : G/Z(G) \rightarrow H/Z(H)$ defined on the set of generators by $\alpha(x_i) = w_i$ gives an isomorphism of $G/Z(G)$ onto $H/Z(H)$. Similarly the map $\beta : \gamma_2(G) \rightarrow \gamma_2(H)$ defined on the set of generators by $\beta(y_{ij}) = z_{ij}$ gives an isomorphism of $\gamma_2(G)$ onto $\gamma_2(H)$. It is not difficult to show that diagram (2.1) commutes in the present setup. Thus it follows that G and H are isoclinic. Hence by Theorem 2.3 and Theorem B we have $P(H) = P(G) = 1$. \square

5. SOME MORE EXAMPLES AND BOUNDS FOR $\text{Pr}_g(G)$

As promised in the introduction, we now show the existence of a finite group G of class 3 such that G has only two different conjugacy class sizes and $|P(G)| > 1$. Consider the

following group for an odd prime p .

$$(5.1) \quad G = \langle x_1, x_2 \mid [x_1, x_2] = y, [x_1, y] = z_1, [x_2, y] = z_2, x_i^p = y^p = z_i^p = 1 (i = 1, 2) \rangle.$$

That G has only two conjugacy class sizes 1 and p^2 , follows from [5, Theorem 4.2]. It is easy to see that the nilpotency class of G is three, $|\gamma_2(G)| = p^3$, $|Z(G)| = p^2$ and $|G| = p^5$.

Let g_1 and g_2 be two elements of G modulo $Z(G)$. Then $g_1 = x_1^{\alpha_1} x_2^{\beta_1} y^{\gamma_1}$ and $g_2 = x_1^{\alpha_2} x_2^{\beta_2} y^{\gamma_2}$ for $0 \leq \alpha_i, \beta_i, \gamma_i \leq p-1$, where $i = 1, 2$. We are now going to calculate $[g_1, g_2]$.

$$(5.2) \quad \begin{aligned} [g_1, g_2] &= [x_1^{\alpha_1} x_2^{\beta_1} y^{\gamma_1}, x_1^{\alpha_2} x_2^{\beta_2} y^{\gamma_2}] = [x_1^{\alpha_1} x_2^{\beta_1}, x_1^{\alpha_2} x_2^{\beta_2}] z_1^{\alpha_1 \gamma_2 - \alpha_2 \gamma_1} z_2^{\beta_1 \gamma_2 - \beta_2 \gamma_1} \\ &= [x_2^{\beta_1}, x_1^{\alpha_2}] [[x_2^{\beta_1}, x_1^{\alpha_2}], x_2^{\beta_2}] [x_1^{\alpha_1}, x_2^{\beta_2}] [[x_1^{\alpha_1}, x_2^{\beta_2}], x_2^{\beta_1}] z_1^{\alpha_1 \gamma_2 - \alpha_2 \gamma_1} z_2^{\beta_1 \gamma_2 - \beta_2 \gamma_1}. \end{aligned}$$

It is not difficult to show that

$$[x_1^{\alpha_2}, x_2^{\beta_1}] = y^{\alpha_2 \beta_1} z_1^{-\beta_1 \alpha_2 (\alpha_2 - 1)/2} z_2^{-\alpha_2 \beta_1 (\beta_1 - 1)/2}.$$

and

$$[x_1^{\alpha_1}, x_2^{\beta_2}] = y^{\alpha_1 \beta_2} z_1^{-\beta_2 \alpha_1 (\alpha_1 - 1)/2} z_2^{-\alpha_1 \beta_2 (\beta_2 - 1)/2}.$$

Putting these values in (5.2), we get

$$(5.3) \quad \begin{aligned} [g_1, g_2] &= y^{\alpha_1 \beta_2 - \alpha_2 \beta_1} z_1^{\beta_1 \alpha_2 (\alpha_2 - 1)/2 - \beta_2 \alpha_1 (\alpha_1 - 1)/2 + \alpha_1 \gamma_2 - \alpha_2 \gamma_1} \\ &\quad z_2^{\alpha_2 \beta_1 (\beta_1 - 1)/2 - \alpha_1 \beta_2 (\beta_2 - 1)/2 - \alpha_1 \beta_1 \beta_2 + \alpha_2 \beta_1 \beta_2 + \beta_1 \gamma_2 - \beta_2 \gamma_1}. \end{aligned}$$

Lemma 5.4. *Let G be the group as defined in (5.1). Then $\text{Pr}_y(G) = \frac{p^2-1}{p^4}$*

Proof. Let $g_1 = x_1^{\alpha_1} x_2^{\beta_1} y^{\gamma_1}$ and $g_2 = x_1^{\alpha_2} x_2^{\beta_2} y^{\gamma_2}$ be two arbitray elements of G modulo the center such that $[g_1, g_2] = y = [x_1, x_2]$. Using (5.3) and comparing powers of y , z_1 and z_2 , we get the following system of equations:

$$(5.5) \quad \alpha_1 \beta_2 - \alpha_2 \beta_1 = 1$$

$$(5.6) \quad \beta_1 \alpha_2 (\alpha_2 - 1)/2 - \beta_2 \alpha_1 (\alpha_1 - 1)/2 + \alpha_1 \gamma_2 - \alpha_2 \gamma_1 = 0$$

$$(5.7) \quad \alpha_2 \beta_1 (\beta_1 - 1)/2 - \alpha_1 \beta_2 (\beta_2 - 1)/2 - \alpha_1 \beta_1 \beta_2 + \alpha_2 \beta_1 \beta_2 + \beta_1 \gamma_2 - \beta_2 \gamma_1 = 0$$

It follows from (5.5) that both of α_1 and β_1 can not be zero. First assume that none of α_1, β_1 is zero. Then substituting the value of β_2 from (5.5) in (5.6) and (5.7), and cancelling out γ_2 using the two new equations, we get

$$\alpha_2 = \frac{\beta_1^2 \alpha_1 + \beta_1 \alpha_1 - \beta_1 + \alpha_1 - 2\gamma_1 - 1}{\beta_1^2}.$$

Now we can find β_2 and γ_2 using (5.5) and (5.6).

Now assume that $\alpha_1 = 0$, $\beta_1 \neq 0$ (or $\beta_1 = 0$, $\alpha_1 \neq 0$). Then, using (5.5)-(5.7), it is not difficult to find α_2, β_2 and γ_2 in terms of β_1 (or α_1) and γ_1 . Thus it follows that given any element $g_1 = x_1^{\alpha_1} x_2^{\beta_1} y^{\gamma_1} z \in G - \gamma_2(G)$, where $z \in Z(G)$, there exists an element $g_2 = x_1^{\alpha_2} x_2^{\beta_2} y^{\gamma_2} \in G$ such that $[g_1, g_2] = y$. Thus $|\{g \in G \mid y \in [g, G]\}| = p^5 - p^3$. Hence by Lemma 3.1, $\text{Pr}_y(G) = \frac{1}{|G|b(G)}(p^5 - p^3) = \frac{p^2-1}{p^4}$, which is the required value for $\text{Pr}_y(G)$. \square

Lemma 5.8. *Let G be the group as defined in (5.1). Then $\text{Pr}_{z_1}(G) = \frac{p^2-1}{p^5}$.*

Proof. Let $g_1 = x_1^{\alpha_1} x_2^{\beta_1} y^{\gamma_1}$ and $g_2 = x_1^{\alpha_2} x_2^{\beta_2} y^{\gamma_2}$ be two arbitray non-trivial elements of G modulo the center such that $[g_1, g_2] = z_1 = [x_1, y]$. Using (5.3) and comparing powers of y , z_1 and z_2 , we get the following system of equations:

$$(5.9) \quad \alpha_1 \beta_2 - \alpha_2 \beta_1 = 0$$

$$(5.10) \quad \beta_1 \alpha_2 (\alpha_2 - 1)/2 - \beta_2 \alpha_1 (\alpha_1 - 1)/2 + \alpha_1 \gamma_2 - \alpha_2 \gamma_1 = 1$$

$$(5.11) \quad \alpha_2 \beta_1 (\beta_1 - 1)/2 - \alpha_1 \beta_2 (\beta_2 - 1)/2 - \alpha_1 \beta_1 \beta_2 + \alpha_2 \beta_1 \beta_2 + \beta_1 \gamma_2 - \beta_2 \gamma_1 = 0$$

We claim that $\beta_1 = 0$. Suppose for a moment that our cliam is true. If $\alpha_1 \neq 0$, then it follows from (5.9) that $\beta_2 = 0$. Substituting $\beta_1 = 0 = \beta_2$ in (5.10) gives

$$(5.12) \quad \alpha_1 \gamma_2 - \alpha_2 \gamma_1 = 1.$$

Notice that $\beta_1 = 0$ and $\beta_2 = 0$ satisfy (5.11). So if we take any element $g_1 = x_1^{\alpha_1} y^{\gamma_1}$ with $\alpha_1 \neq 0$, then there exists $g_2 \in G$ such that $[g_1, g_2] = z_1$, where $g_2 = x_1^{\alpha_2} y^{\gamma_2}$ in which $0 \leq \alpha_2, \gamma_2 \leq p-1$ satisfy (5.12). If $\alpha_1 = 0$, then $\gamma_1 \neq 0$ as g_1 is a non-trivial element. Substituting $\alpha_1 = \beta_1 = 0$ in (5.10) and (5.11), we respectively get

$$\gamma_1 \alpha_2 = -1 \quad \text{and} \quad \gamma_1 \beta_2 = 0.$$

Since $\gamma_1 \neq 0$, we have $\beta_2 = 0$. So if we take any element $g_1 = y^{\gamma_1}$ with $\gamma_1 \neq 0$, then there exists $g_2 \in G$ such that $[g_1, g_2] = z_1$, where $g_2 = x_1^{\alpha_2}$ in which $0 \leq \alpha_2 \leq p-1$ satisfies $\gamma_1 \alpha_2 = -1$. Hence it follows that for any non-central element $g_1 = x_1^{\alpha_1} y^{\gamma_1} z$, where $z \in Z(G)$, there exists an element $g_2 \in G$ such that $[g_1, g_2] = z_1$. Thus $|\{g \in G \mid z_1 \in [g, G]\}| = p^4 - p^2$. Hence by Lemma 3.1, $\text{Pr}_{z_1}(G) = \frac{1}{|G|b(G)}(p^4 - p^2) = \frac{p^2-1}{p^5}$, which is the required value for $\text{Pr}_{z_1}(G)$.

Now we prove our claim. First suppose that $\alpha_1 \neq 0$. Assume contrarily that $\beta_1 \neq 0$. Then substituting the value of β_2 from (5.9) in (5.10) and (5.11), and cancelling out γ_2 using the two new equations, we get $2\beta_1 = 0$. Since $\beta_1 \neq 0$ and p is odd, this is not possible. Now assume that $\alpha_1 = 0$ and $\beta_1 \neq 0$. It follows from (5.9) that $\alpha_2 = 0$. Substituting $\alpha_1 = \alpha_2 = 0$, in (5.10), we get $0 = 1$, which is absurd. Hence $\beta_1 = 0$, which proves our claim as well as the lemma. \square

The following result follows from the preceding two lemmas.

Theorem 5.13. *Let G be the group as defined in (5.1). Then $|P(G)| > 1$.*

Now we calculate $\text{Pr}_g(G)$ for finite Camina p -groups of class 3.

Proposition 5.14. *Let G be a finite Camina p -group of class 3. Then*

$$\text{Pr}_g(G) = \begin{cases} \frac{1}{|G|} \left(\frac{|G-\gamma_2(G)|}{|\gamma_2(G)|} + \frac{|\gamma_2(G)-\gamma_3(G)|}{|\gamma_3(G)|} \right) & \text{if } g \in \gamma_3(G) \\ \text{Pr}_g(G) = \frac{1}{|G|} \frac{|G-\gamma_2(G)|}{|\gamma_2(G)|} & \text{if } g \in \gamma_2(G) - \gamma_3(G). \end{cases}$$

Moreover, $P(G) = 2$.

Proof. Notice that $K(G) = \gamma_2(G)$ for a finite Camina p -group G . Also if G is a finite Camina p -group of class 3, then for $g \in \gamma_2(G) - \gamma_3(G)$, $\{x \in G \mid g \in [x, G]\} = G - \gamma_2(G)$, for $g \in \gamma_3(G)$, $\{x \in G \mid g \in [x, G]\} = G - \gamma_3(G)$ and

$$|x^G| = \begin{cases} |\gamma_3(G)| = |Z(G)| & \text{if } x \in \gamma_2(G) - \gamma_3(G) \\ |\gamma_2(G)| & \text{if } x \in G - \gamma_2(G). \end{cases}$$

If $1 \neq g \in \gamma_3(G)$, then by Lemma 3.1 and the above information we have

$$\Pr_g(G) = \frac{1}{|G|} \sum_{g \in [x, G]} \frac{1}{|x^G|} = \frac{1}{|G|} \left(\frac{|G - \gamma_2(G)|}{|\gamma_2(G)|} + \frac{|\gamma_2(G) - \gamma_3(G)|}{|\gamma_3(G)|} \right).$$

If $g \in \gamma_2(G) - \gamma_3(G)$, then again by Lemma 3.1 and the above information we have

$$\Pr_g(G) = \frac{1}{|G|} \sum_{g \in [x, G]} \frac{1}{|x^G|} = \frac{1}{|G|} \frac{|G - \gamma_2(G)|}{|\gamma_2(G)|}.$$

Obviously $P(G) = 2$, since $\Pr_g(G)$ takes only two values for all non-trivial $g \in \gamma_2(G)$. This completes the proof. \square

Now we discuss some bounds on $\Pr_g(G)$. As a simple consequence of Lemma 3.1, we get the following result, which also gives a relationship between $\Pr_g(G)$ and $\Pr(G)$.

Lemma 5.15. *Let G be a finite group. Then for $1 \neq g \in K(G)$, we have*

$$(5.16) \quad \frac{1}{|G|b(G)} |\{x \in G \mid g \in [x, G]\}| \leq \Pr_g(G) \leq \frac{1}{|G|} \sum_{x \in G - Z(G)} \frac{1}{|x^G|}.$$

Moreover,

- (i) equality holds on the right side if and only if $g \in [x, G]$ for all $x \in G - Z(G)$,
- (ii) equality holds on the left side if and only if $|x^G| = b(G)$ for all $x \in G$ such that $g \in [x, G]$,
- (iii) $\Pr_g(G) < \Pr(G)$ for all $1 \neq g \in \gamma_2(G)$.

Proof. Let $1 \neq g \in K(G)$. It easily follows from Lemma 3.1 that

$$\Pr_g(G) = \frac{1}{|G|} \sum_{g \in [x, G]} \frac{1}{|x^G|} \leq \frac{1}{|G|} \sum_{x \in G - Z(G)} \frac{1}{|x^G|}.$$

By comparing summation terms, notice that equality holds on the right side of (5.16) if and only if $g \in [x, G]$ for all $x \in G - Z(G)$. So (i) holds true. Again using Lemma 3.1, we have

$$\Pr_g(G) = \frac{1}{|G|} \sum_{g \in [x, G]} \frac{1}{|x^G|} \geq \frac{1}{|G|} \sum_{g \in [x, G]} \frac{1}{|b(G)|} = \frac{1}{|G|b(G)} |\{x \in G \mid g \in [x, G]\}|.$$

It is again obvious to see that (ii) holds true.

Finally

$$\frac{1}{|G|} \sum_{x \in G - Z(G)} \frac{1}{|x^G|} < \frac{1}{|G|} \left(|Z(G)| + \sum_{x \in G - Z(G)} \frac{1}{|x^G|} \right) = \frac{1}{|G|} \sum_{x \in G} \frac{1}{|x^G|} = \Pr(G).$$

Hence, for $1 \neq g \in \gamma_2(G)$, $\Pr_g(G) < \Pr(G)$. \square

Notice that in the preceding lemma equality hold simultaneously on both sides of (5.16) if there exists an element $1 \neq g' \in K(G)$ such that $g' \in [x, G]$ for all $x \in G - Z(G)$ and $|x^G| = b(G)$ for all such x . For such an element g' , $\Pr_{g'}(G) = \frac{1}{|G|b(G)} |G - Z(G)| = \frac{1}{|b(G)|} (1 - \frac{1}{|G:Z(G)|})$. Let us set $B(G) = \frac{1}{|b(G)|} (1 - \frac{1}{|G:Z(G)|})$. Perhaps nothing special can be said about the relationship between $\Pr_g(G)$ and $B(G)$ for various $1 \neq g \in K(G)$. This relationship highly depends on the given group G as well as on the non-trivial element $g \in K(G)$. For example, if we consider a finite Camina p -group of nilpotency class 3, then it follows from Proposition 5.14 that $\Pr_g(G) > B(G)$ if $g \in \gamma_3(G)$ and $\Pr_g(G) < B(G)$ if $g \in \gamma_2(G) - \gamma_3(G)$. It is not

difficult to show that $\text{Pr}_g(G) < B(G)$, for each $1 \neq g \in K(G)$, for the group G defined in (5.1). Also if $\text{Pr}_g(G) = B(G)$ for some group G and some $1 \neq g \in K(G)$, we do not know what interesting can be said for the group G itself, except the fact that such groups will have only two different conjugacy class sizes. But if, for some group G , $\text{Pr}_g(G) = B(G)$ for all $1 \neq g \in K(G)$, then we have the following nice characterization of such group.

Theorem 5.17. *Let G be a finite group. Then $\text{Pr}_g(G) = B(G)$ for all $1 \neq g \in K(G)$ if and only if G is isoclinic to a finite Camina special p -group for some prime integer p .*

Proof. Suppose that $\text{Pr}_g(G) = B(G)$ for all $1 \neq g \in K(G)$. Then it follows from the preceding lemma that $|x^G| = b(G)$ for all $x \in G - Z(G)$. Thus G has only two different conjugacy class sizes. Now it follows from Theorem 2.6 that G is isoclinic to a finite p -group H (say) for some prime integer p , and the nilpotency class of H is either 2 or 3. So let us work with H now. Notice that $B(G) = B(H)$. Thus $\text{Pr}_h(H) = B(H)$ for all $1 \neq h \in K(H)$ and therefore $|P(H)| = 1$. Let $1 \neq h \in K(H)$. Then $h \in [y, H]$ for all $y \in H - Z(H)$, showing that $h \in \cap_{y \in H - Z(H)} [y, H]$. Since this is true for each $1 \neq h \in K(H)$, we have $K(H) \in \cap_{y \in H - Z(H)} [y, H]$. This is possible only when $[y, H] = K(H)$ for all $y \in H - Z(H)$, since $[y, H] \subseteq K(H)$ for all $y \in H$. We know that the nilpotency class of H is either 2 or 3. We claim that H is of class 2. Assume that the nilpotency class of H is 3. Then there exists an element $u \in \gamma_2(G) - Z(G)$. By what we have, it follows that $K(H) = [u, H]$. Thus $\gamma_2(H) = \langle K(H) \rangle \leq \gamma_3(H)$, since $[u, H] \subseteq \gamma_3(H)$. This contradiction proves our claim that the nilpotency class of H is 2. Notice that in a finite p -group X of class 2, for any $x \in X$, $[x, X]$ is a subgroup of $\gamma_2(X)$. Since $K(H) = [y, H]$ for any element $y \in H - Z(H)$ and $[y, H]$ is a subgroup of $\gamma_2(H)$, it follows that $\gamma_2(H) = \langle K(H) \rangle = [y, H] \leq \gamma_2(H)$. Hence $[y, H] = \gamma_2(H)$ for all $y \in H - Z(H)$. It now follows from Proposition 2.4, the definition of Camina groups and Theorem 2.7 that H is isoclinic to a Camina special p -group.

If part of the theorem follows from Theorem A. □

We conclude this section with the following simple minded application of commuting probability. The following result follows from [11, Proposition 1], which is proved using degree equation. Since the proof is as nice as eating grapes, we did not make efforts to re-produce a character free proof.

Proposition 5.18. *Let G be a non-abelian finite group and p be the smallest prime dividing $|G|$. Then $\text{Pr}(G) > \frac{1}{p}$ if and only if G is isoclinic to an extraspecial finite p -group.*

Notice that any extraspecial p -group has only two different class sizes, namely, 1 and p . Thus it follows that if G is any group which is isoclinic to an extraspecial p -group, then G has only two different class sizes 1 and p . The following result shows that the converse also holds true.

Theorem 5.19. *Let G be a finite group which has only two conjugacy class sizes 1 and p , where p is any prime integer. Then G is isoclinic to an extraspecial finite p -group.*

Proof. Let G be a finite group having only two conjugacy class sizes 1 and p , where p is any prime integer. Then by Theorem 2.6, G is isoclinic to a finite p -group H (say), and therefore $b(H) = b(G) = p$. Now it follows from Proposition 3.2 that $\text{Pr}(G) = \text{Pr}(H) > 1/p$. Thus by Proposition 5.18, H is isoclinic to a finite extraspecial p -group. This completes the proof of the theorem. □

Remark 5.20. As mentioned in the introduction and as observed by Ishikawa [5], Theorem 5.19 can be easily obtained by using the following deep result of Vaughan-Lee [9, Main Theorem], which states that for any finite p -group G , $|\gamma_2(G)| \leq p^{\frac{b(G)(b(G)+1)}{2}}$.

Acknowledgements. The authors thank Pradeep K. Rai for pointing out a gap in the proof of Lemma 4.3 in an old version of the paper.

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